Estimating Cross-Industry Cross-Country Interaction Models Using Benchmark Industry Characteristics

Appendix

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May 2019

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1 Detailed Derivation of Equation (8) in the Main Text

Using (2) in (1) in the main text yields that the demeaned outcome in the numerator of (7) can be written as $y_{in} - \overline{y}_i - \overline{y}_n + \overline{y} = \beta(z_i - \overline{z})(x_n - \overline{x}) + \upsilon_{in}$. Here z_i is the global technological industry characteristic of industry i, $\overline{z}$ is the average technological industry characteristic across all industries, and

$$\upsilon_{in} = u_{in} - \overline{u}_n - \overline{u}_i + \overline{u} \tag{A1}$$

with

$$u_{in} = (\alpha + \beta x_n)\varepsilon_{in}, \tag{A2}$$

where $\overline{u}_n$ is the average of u_{in} across industries i for country n, $\overline{u}_i$ is the average of u_{in} across countries n for industry i, and $\overline{u}$ is the average of u_{in} both across countries and across industries. Substituting $y_{in} - \overline{y}_i - \overline{y}_n + \overline{y} = \beta(z_i - \overline{z})(x_n - \overline{x}) + \upsilon_{in}$ in (7) yields

$$\widehat{b} = \beta \frac{\frac{1}{I}\sum_{i=1}^{I}(z_{iUS} - \overline{z}_{US})(z_i - \overline{z})}{\frac{1}{I}\sum_{i=1}^{I}(z_{iUS} - \overline{z}_{US})^2} + \frac{\frac{1}{N}\frac{1}{I}\sum_{n=1}^{N}\sum_{i=1}^{I}(z_{iUS} - \overline{z}_{US})(x_n - \overline{x})\upsilon_{in}}{\frac{1}{N}\frac{1}{I}\sum_{n=1}^{N}\sum_{i=1}^{I}(z_{iUS} - \overline{z}_{US})^2(x_n - \overline{x})^2}. \tag{A3}$$

Note that the first ratio on the right-hand side of (A3) does not involve $\frac{1}{N}\sum_{n=1}^{N}(x_n - \overline{x})^2$ as this term cancels out.

Using (2), we can write demeaned US industry characteristics in terms of global and US-specific industry characteristics: $z_{iUS} - \overline{z}_{US} = (z_i - \overline{z}) + (\varepsilon_{iUS} - \overline{\varepsilon}_{US})$. Substituting in (A3) yields

$$\widehat{b} = \beta \frac{\frac{1}{I}\sum_{i=1}^{I}(z_i - \overline{z})^2 + \frac{1}{I}\sum_{i=1}^{I}(z_i - \overline{z})(\varepsilon_{iUS} - \overline{\varepsilon}_{US})}{\frac{1}{I}\sum_{i=1}^{I}(z_{iUS} - \overline{z}_{US})^2} + \frac{\frac{1}{N}\frac{1}{I}\sum_{n=1}^{N}\sum_{i=1}^{I}(z_i - \overline{z})(x_n - \overline{x})\upsilon_{in}}{\frac{1}{N}\sum_{n=1}^{N}(x_n - \overline{x})^2 \frac{1}{I}\sum_{i=1}^{I}(z_{iUS} - \overline{z}_{US})^2} \tag{A4}$$

$$+ \frac{\frac{1}{N}\frac{1}{I}\sum_{n=1}^{N}\sum_{i=1}^{I}(\varepsilon_{iUS} - \overline{\varepsilon}_{US})(x_n - \overline{x})\upsilon_{in}}{\frac{1}{N}\sum_{n=1}^{N}(x_n - \overline{x})^2 \frac{1}{I}\sum_{i=1}^{I}(z_{iUS} - \overline{z}_{US})^2}.$$

We will now discuss the probability limit as I goes to infinity of each of the three ratios on the right-hand side of (A4).

To begin with, we show that the probability limit of the first ratio on the right-hand side of (A4) is $\beta(1 - \phi)$. To see this note that the second term in the numerator can be written as $\frac{1}{I}\sum_{i=1}^{I}(z_i-\overline{z})(\varepsilon_{iUS}-\overline{\varepsilon}_{US})=\frac{1}{I}\sum_{i=1}^{I}z_i\varepsilon_{iUS}-\overline{z}\frac{1}{I}\sum_{i=1}^{I}\varepsilon_{iUS}$. As z_i is i.i.d., the standard version of the law of large numbers yields that the probability limit of $\overline{z}$ is $E(z_i)$. Using the law of large numbers for independent random variables with the same expectation and bounded variances we obtain probability limits for the two averages across industries, $\frac{1}{I}\sum_{i=1}^{I}z_i\varepsilon_{iUS}$ and $\frac{1}{I}\sum_{i=1}^{I}\varepsilon_{iUS}$. The probability limit of the first average is equal to $Ez_i\varepsilon_{iUS}=Ez_iE\varepsilon_{iUS}=0$, as z_i is independent of all other model elements and $E\varepsilon_{iUS}=0$. The probability limit of the second average is $E\varepsilon_{iUS}=0$. Thus, $\frac{1}{I}\sum(z_i-\overline{z})(\varepsilon_{iUS}-\bar{\varepsilon}_{US})$ vanishes in the limit as I goes to infinity. Moreover, the probability limitsof $\frac{1}{I}\sum_{i=1}^{I}(z_{iUS}-\overline{z}_{US})^2$ and $\frac{1}{I}\sum_{i=1}^{I}(z_i-\overline{z})^2$ are $Var(z_{US})$ and $Var(z_i)$ respectively. Eventually, (5) in the main text implies $1-\phi=Var(z_i)/Var(z_{US})$.

Next, we show that the probability limit, as I goes to infinity, of the second ratio on the right-hand side of (A4) is zero. Using (A1), the numerator of this ratio can be written as

$$\frac{1}{N}\sum_{n=1}^{N}(x_n-\overline{x})\left[\frac{1}{I}\sum_{i=1}^{I}(z_i-\overline{z})(u_{in}-\overline{u}_n-\overline{u}_i+\overline{u})\right] \tag{A5}$$

and the square bracket can be written as

$$\frac{1}{I}\sum_{i=1}^{I}z_i(u_{in}-\overline{u}_i)-\overline{z}\frac{1}{I}\sum_{i=1}^{I}(u_{in}-\overline{u}_i)-(\overline{u}_n-\overline{u})\frac{1}{I}\sum_{i=1}^{I}z_i+\overline{z}(\overline{u}_n-\overline{u}). \tag{A6}$$

All weighted sums across industries in (A6) are sums of independent random variables with equal expectation and bounded variances. Hence, the law of large numbers implies that the probability limit of the first weighted sum is $Ez_i(u_{in}-\overline{u}_i)=Ez_iE(u_{in}-\overline{u}_i)=0$, where we use that global industry characteristics z_i are independent of all other model elements and that $E(u_{in}-\overline{u}_i)=Eu_{in}-E\overline{u}_i=0$. The probability limits of the second and third weighted sum are $E(u_{in}-\overline{u}_i)=Eu_{in}-E\overline{u}_i=0$ and Ez_i respectively. Again, as z_i is i.i.d., the probability limit of $\overline{z}$ is $E(z_i)$. Moreover, the terms $\overline{u}_n$ and $\overline{u}$ in (A6) go to zero in probability, as $E\overline{u}_n=E\overline{u}=0$ and the variances $Var(\overline{u}_n)=\frac{1}{I}(\alpha+\beta x_n)^2\sigma^2$ and $Var(\overline{u})=Var(\frac{1}{I}\sum_{i=1}^{I}\overline{u}_i)=\frac{1}{I}Var(\overline{u}_i)=\frac{1}{I}\frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N}(\alpha+\beta x_n)(\alpha+\beta x_m)\rho_{nm}\sigma^2$ go to zero as I goes to infinity. Hence, all terms in (A6) vanish in the limit. At the same time, the denominator of the second ratio in (A4) goes to some strictly positive number as I goes to infinity. On the other hand, the denominator of this ratio goes to some strictly positive number as I goes to infinity. Hence, the second ratio in (A4) vanishes in the probability limit.

Collecting the results we have so far, as I goes to infinity, the probability limit of (A4) is

$$b = (1-\phi)\beta + \frac{\frac{1}{N}\sum_{n=1}^{N}(x_n - \overline{x}) \operatorname*{plim}_{I\to\infty} \frac{1}{I}\sum_{i=1}^{I}(\varepsilon_{iUS} - \overline{\varepsilon}_{US})\upsilon_{in}}{Var(z_{US})\frac{1}{N}\sum_{n=1}^{N}(x_n - \overline{x})^2} \tag{A7}$$

where we rewrote the numerator of the last term in (A4) in terms of an outer sum across countries and an inner sum across industries. The key term in (A7) is the term behind the probability limit (plim). Using (A1), this term can be written as

$$\frac{1}{I}\sum_{i=1}^{I}(\varepsilon_{iUS} - \overline{\varepsilon}_{US})(u_{in} - \overline{u}_i) - (\overline{u}_n - \overline{u})\frac{1}{I}\sum_{i=1}^{I}(\varepsilon_{iUS} - \overline{\varepsilon}_{US}). \tag{A8}$$

The second term in (A8) is equal to zero, as $\overline{\varepsilon}_{US} = \frac{1}{I}\sum_{i=1}^{I}\varepsilon_{iUS}$. The first term can be written as

$$\frac{1}{I}\sum_{i=1}^{I}(\varepsilon_{iUS} - \overline{\varepsilon}_{US})(u_{in} - \overline{u}_i) = \frac{1}{I}\sum_{i=1}^{I}\varepsilon_{iUS}(u_{in} - \overline{u}_i) - \overline{\varepsilon}_{US}(\overline{u}_n - \overline{u}). \tag{A9}$$

As $E\bar{\varepsilon}_{US} = E\bar{u}_n, = E\bar{u} = 0$ and the variances $Var(\overline{\varepsilon}_{US}) = \frac{1}{I}\sigma^2$, $Var(\overline{u}_n) = \frac{1}{I}(\alpha + \beta x_n)^2\sigma^2$, and $Var(\overline{u}) = Var(\frac{1}{I}\sum_{i=1}^{I}\overline{u}_i) = \frac{1}{I}Var(\overline{u}_i) = \frac{1}{I}\frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N}(\alpha + \beta x_n)(\alpha + \beta x_m)\rho_{nm}\sigma^2$ vanish as I tends towards infinity, the second term on the right-hand side of (A9) goes to zero in probability. Making use of the law of large numbers for independent random variables of equal expectation and bounded variance the probability limit of the first term on the right-hand side of (A9) is

$$E\varepsilon_{iUS}(u_{in} - \overline{u}_i) = (\alpha + \beta x_n)E\varepsilon_{iUS}\varepsilon_{in} - \frac{1}{N}\sum_{n=1}^{N}(\alpha + \beta x_n)E\varepsilon_{iUS}\varepsilon_{in}. \tag{A10}$$

Noting that $\sigma^2\rho_{nUS} = E\varepsilon_{iUS}\varepsilon_{in}$, we have

$$E\varepsilon_{iUS}(u_{in} - \overline{u}_i) = (\alpha + \beta x_n)\sigma^2\rho_{nUS} - \frac{1}{N}\sum_{n=1}^{N}(\alpha + \beta x_n)\sigma^2\rho_{nUS}. \tag{A11}$$

Using this, the numerator of the second term on the right-hand side of (A7) is

$$\frac{1}{N}\sum_{n=1}^{N}(x_n - \overline{x})(\alpha+\beta x_n)\sigma^2\rho_{nUS} - \left(\frac{1}{N}\sum_{n=1}^{N}(\alpha+\beta x_n)\sigma^2\rho_{nUS}\right)\frac{1}{N}\sum_{n=1}^{N}(x_n - \overline{x}). \tag{A12}$$

As $\frac{1}{N}\sum_{n=1}^{N} x_n = \overline{x}$, the second term in (A12) is zero. Substituting the first term in (A12) for the numerator in (A7) yields

$$b = (1-\phi)\beta + \left(\frac{\sigma^2}{Var(z_{US})}\right)\frac{\frac{1}{N}\sum_{n=1}^{N}(x_n - \overline{x})(\alpha+\beta x_n)\rho_{nUS}.}{\frac{1}{N}\sum_{n=1}^{N}(x_n - \overline{x}^2)}. \tag{A13}$$

Using the definitions for A from (9) and for B from (10) in the main text as well as the fact that $\phi = \sigma^2/(\sigma^2 + Var(z_i)) = \sigma^2/Var(z_{US})$, rewriting (A13) yields (8).

2 Detailed Derivation of Equation (36) in the Main Text

We are interested in the probability limit of $\frac{1}{I}\sum_{i=1}^{I}\widehat{u}_{in}\widehat{u}_{im}$ as the number of industries I goes to infinity, where

$$\widehat{u}_{in} = \upsilon_{in} - (x_n - \overline{x})\sum_{k=1}^{N}\psi_k\upsilon_{ik}, \tag{A14}$$

ψ_k is the least-squares regression weight defined in (35) in the main text, and

$$\upsilon_{in} = u_{in} - \overline{u}_n - \overline{u}_i + \overline{u}. \tag{A15}$$

In (A15), $\overline{u}_n$ is the average of u_{in} across industries i for country n, $\overline{u}_i$ is the average of u_{in} across countries n for industry i, and $\overline{u}$ is the average of u_{in} both across countries and across industries. Making use of (A14),

$$\frac{1}{I}\sum_{i=1}^{I}\widehat{u}_{in}\widehat{u}_{im} = \frac{1}{I}\sum_{i=1}^{I}\upsilon_{in}\upsilon_{im} - (x_n - \overline{x})\sum_{k=1}^{N}\psi_k\left(\frac{1}{I}\sum_{i=1}^{I}\upsilon_{ik}\upsilon_{im}\right) - (x_m - \overline{x})\sum_{k=1}^{N}\psi_k\left(\frac{1}{I}\sum_{i=1}^{I}\upsilon_{ik}\upsilon_{in}\right) + (x_n - \overline{x})(x_m - \overline{x})\sum_{k=1}^{N}\sum_{g=1}^{N}\psi_g\psi_k\left(\frac{1}{I}\sum_{i=1}^{I}\upsilon_{in}\upsilon_{im}\right). \tag{A16}$$

As (A16) reveals, a key term to determine the probability limit of $\frac{1}{I}\sum_{i=1}^{I}\widehat{u}_{in}\widehat{u}_{im}$ is the probability limit of

$$\frac{1}{I}\sum_{i=1}^{I}\upsilon_{in}\upsilon_{im}. \tag{A17}$$

We show that this probability limit is $\omega_{nm}-\overline{\omega}_n-\overline{\omega}_m+\overline{\omega}$, where ω_{nm} is the covariance $Eu_{in}u_{im}$ defined in (23) in the main text, $\overline{\omega}_p$ denotes the average of ω_{pq} across q, i.e. $\overline{\omega}_p = \frac{1}{N}\sum_{q=1}^{N}\omega_{pq}$, and $\overline{\omega}$ is the average of ω_{pq} across q and p, i.e. $\overline{\omega} = \frac{1}{N^2}\sum_{p=1}^{N}\sum_{q=1}^{N}\omega_{pq}$. To this end, it is useful to use (A1) to rewrite (A17) as the weighted sum of four terms:

$$\frac{1}{I}\sum_{i=1}^{I}\upsilon_{in}\upsilon_{im} = \frac{1}{I}\sum_{i=1}^{I}(u_{in}-\overline{u}_i)(u_{im}-\overline{u}_i) + (\overline{u}_n-\overline{u})(\overline{u}_m-\overline{u}) - (\overline{u}_m-\overline{u})\frac{1}{I}\sum_{i=1}^{I}(u_{in}-\overline{u}_i) - (\overline{u}_n-\overline{u})\frac{1}{I}\sum_{i=1}^{I}(u_{im}-\overline{u}_i). \tag{A18}$$

All $(\overline{u}_n-\overline{u})$-terms on the right-hand side of (A18) go to zero in probability as the number of industries I goes to infinity. To see this, note that $E(\overline{u}_n-\overline{u}) = 0$ and that the variance $Var(\overline{u}_n-\overline{u})$ goes to zero as the number of industries I goes to infinity. This can be verified by writing the variance as

$$E(\overline{u}_n-\overline{u})^2 = E\overline{u}_n^2 - 2E\overline{u}_n\overline{u} + E\overline{u}^2. \tag{A19}$$

Now, the three terms on the right-hand side of (A19) can be respectively written as

$$E\overline{u}^2 = E\left(\frac{1}{I}\sum_{i}^{I}\overline{u}_i\right)^2 = \frac{1}{I}E\overline{u}_i^2 = \frac{1}{I}\frac{1}{N^2}\sum_{g=1}^{N}\sum_{k=1}^{N}\omega_{gk}, \tag{A20}$$

$$E\overline{u}_n^2 = E\left(\frac{1}{I}\sum_{j=1}^{I}u_{jn}\right)^2 = \frac{1}{I}\omega_{nn}, \tag{A21}$$

and

$$2E\overline{u}_n\overline{u} = 2\frac{1}{N}\sum_{k=1}^{N}E\overline{u}_n\overline{u}_k = 2\frac{1}{N}\frac{1}{I}\sum_{k=1}^{N}\omega_{nk}. \tag{A22}$$

Therefore, all three terms vanish in the limit as the number of industries I goes to infinity.

The terms on the right-hand side of (A18) that involve weighted sums across industries can be analyzed using the law of large numbers for independent random variables with the same expectation and bounded variances. Thus, the probability limit of

$$\frac{1}{I}\sum_{i=1}^{I}(u_{im} - \overline{u}_i) \tag{A23}$$

is $E(u_{im} - \overline{u}_i) = Eu_{im} - E\overline{u}_i = 0$. Combined with the properties of the term $\overline{u}_n - \overline{u}$ discussed in (A19)–(A22), this implies that the probability limit of all terms on the right-hand side of (A18) except the first one is zero. By another application of the law of large numbers for independent random variables of equal expectation and bounded variance implies, the probability limitof

$$\frac{1}{I}\sum_{i=1}^{I}(u_{in} - \overline{u}_i)(u_{im} - \overline{u}_i) \tag{A24}$$

is $E(u_{in} - \overline{u}_i)(u_{im} - \overline{u}_i)$, which can be further calculated to be

$$E(u_{in} - \overline{u}_i)(u_{im} - \overline{u}_i) = \omega_{nm} - \overline{\omega}_n - \overline{\omega}_m + \overline{\omega}. \tag{A25}$$

Hence, it follows that, as the number of industries I goes to infinity, the probability limit of $\frac{1}{I}\sum_{i=1}^{I} \upsilon_{in}\upsilon_{im}$ is $\omega_{nm} - \overline{\omega}_n - \overline{\omega}_m + \overline{\omega}$.

Returning to the analysis of (A16), we have just shown the probability limit of the first term to be equal to

$$\omega_{km} - \overline{\omega}_k - \overline{\omega}_m + \overline{\omega} \tag{A26}$$

as the number of industries I tends to infinity. The probability limit of the second term in (A16) is

$$(x_n - \overline{x})\sum_{k=1}^{N}\psi_k(\omega_{km} - \overline{\omega}_k - \overline{\omega}_m + \overline{\omega}) = (x_n - \overline{x})\sum_{k=1}^{N}\psi_k(\omega_{km} - \overline{\omega}_k) \tag{A27}$$

where we have once again substituted $\omega_{nm} - \overline{\omega}_n - \overline{\omega}_m + \overline{\omega}$ for the probability limit of $\frac{1}{I}\sum_{i=1}^{I} \upsilon_{in}\upsilon_{im}$ and made use of $\sum_{k=1}^{N}\psi_k = 0$. The probability limit of the third term in (A16) is equal to (A27) with n and m switched. Finally, the probability limit of the last term in (A16) is

$$
\begin{aligned}
&(x_m - \overline{x})(x_n - \overline{x}) \sum_{k=1}^{N} \sum_{g=1}^{N} \psi_g \psi_k (\omega_{kg} - \overline{\omega}_k - \overline{\omega}_g + \overline{\omega}) \\
=&(x_m - \overline{x})(x_n - \overline{x}) \sum_{k=1}^{N} \sum_{g=1}^{N} \psi_g \psi_k \omega_{kg},
\end{aligned} \tag{A28}
$$

where we made use of $\sum_{k=1}^{N} \psi_k = 0$ again. Collecting the results in (A26)-(A28) yields that, as the number of industries I goes to infinity, the probability limit of $\frac{1}{I} \sum_{i=1}^{I} \widehat{u}_{in} \widehat{u}_{im}$ is

$$
\begin{aligned}
\omega_{nm} - \overline{\omega}_n - \overline{\omega}_m + \overline{\omega} - (x_m - \overline{x}) &\sum_{k=1}^{N} \psi_k (\omega_{kn} - \overline{\omega}_k) \\
- (x_n - \overline{x}) &\sum_{k=1}^{N} \psi_k (\omega_{km} - \overline{\omega}_k) \\
+ (x_m - \overline{x})(x_n - \overline{x}) &\sum_{k=1}^{N} \sum_{g=1}^{N} \psi_g \psi_k \omega_{kg}.
\end{aligned} \tag{A29}
$$

Defining

$$
\mu_n = \overline{\omega}_n - \frac{1}{2} \overline{\omega} \tag{A30}
$$

$$
\lambda_n = \sum_{k=1}^{N} \psi_k (\omega_{kn} - \overline{\omega}_k) - \frac{1}{2} (x_n - \overline{x}) \sum_{k=1}^{N} \sum_{g=1}^{N} \psi_g \psi_k \omega_{kg} \tag{A31}
$$

(A29) can be rewritten as

$$
\omega_{nm} - \mu_n - \mu_m - (x_m - \overline{x}) \lambda_n - (x_n - \overline{x}) \lambda_m \tag{A32}
$$

which is the right-hand side of (36) in the main text.

It remains to be shown that, as claimed in the main text, $\sum_{n=1}^{N} \lambda_n = 0$. However, this follows immediately from the fact that $\frac{1}{N} \sum_{n=1}^{N} x_n = \overline{x}$ and $\frac{1}{N} \sum_{n=1}^{N} \omega_{kn} = \overline{\omega}_k$.

3 Show that Equation (36) in the Main Text Does Not Determine ω_{nm} for Arbitrary $\mathbf{\Omega}$

Using standard results in econometrics it can be shown that it is impossible to identify the elements ω_{nm} from the π_{nm} in (36) in the main text for an arbitrary variance-covariance matrix $\mathbf{\Omega}$. To do so, we collect the π_{nm} in a $N \times N$ matrix $\mathbf{\Pi}$ and note that the equation system in (36) can be rewritten in matrix form as

$$\mathbf{\Pi} = \mathbf{M\Omega M} \tag{A33}$$

where $\mathbf{M} = \mathbf{I} - \mathbf{P}$, $\mathbf{I}$ is a square identity matrix of size N, $\mathbf{P}$ is the projection matrix $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, and $\mathbf{X} = (1, \mathbf{x})$ with 1 being a column vector of length N and $\mathbf{x}' = (x_1, \ldots, x_N)$. The key issue then becomes whether the equation system in (A33) determines the symmetric variance-covariance matrix $\mathbf{\Omega}$ for given $\mathbf{\Pi}$ and $\mathbf{M}$. Using the fact that $\mathbf{P}$ is a projection matrix, i.e. $\mathbf{PX} = \mathbf{X}$ and thus $\mathbf{MX} = 0$, it is easy to show that $\mathbf{\Omega}$ cannot be determined. Indeed, if $\mathbf{\Omega}$ solves (A33) then so does any $\widetilde{\mathbf{\Omega}} = \mathbf{\Omega} + \mathbf{XD} + \mathbf{D}'\mathbf{X}' + \mathbf{XEE}'\mathbf{X}'$, where $\mathbf{D}$ and $\mathbf{E}$ are arbitrary $2 \times N$ matrices. Hence, (A33) does not identify $\mathbf{\Omega}$.

Next, we verify that equation (36) can indeed be rewritten as $\mathbf{\Pi} = \mathbf{M\Omega M}$. Using the definitions introduced above, we can rewrite $\mathbf{\Pi} = \mathbf{M\Omega M}$ as

$$\begin{aligned} \mathbf{\Pi} = \mathbf{\Omega} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Omega} - \mathbf{\Omega}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' \\ + \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Omega}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'. \end{aligned} \tag{A34}$$

The first step to show that this corresponds to (36) in the main text is to write $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$ as

$$\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} = \left(\sum_{k=1}^{N}(x_k - \overline{x})^2\right)^{-1} \begin{pmatrix} \frac{1}{N}\sum_{k=1}^{N} x_k^2 - x_1\overline{x} & x_1 - \overline{x} \\ \vdots & \vdots \\ \frac{1}{N}\sum_{k=1}^{N} x_k^2 - x_N\overline{x} & x_N - \overline{x} \end{pmatrix} \tag{A35}$$

and $\mathbf{X}'\mathbf{\Omega}$ as

$$\mathbf{X}'\mathbf{\Omega} = \begin{pmatrix} N\overline{\omega}_1 & \ldots & N\overline{\omega}_N \\ \sum_{k=1}^{N} x_k\omega_{k1} & \ldots & \sum_{k=1}^{N} x_k\omega_{kN} \end{pmatrix}, \tag{A36}$$

where ω_{nm} is the typical element of $\mathbf{\Omega}$ and $\overline{\omega}_p$ denotes the average of ω_{pq} across q. Hence the typical element of the matrix $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Omega}$ in (A34) is

$$\left(\sum_{k=1}^{N}(x_k - \overline{x})^2\right)^{-1} \left[\left(\sum_{k=1}^{N} x_k^2 - N\overline{x}^2\right)\overline{\omega}_m - (x_n - \overline{x})\overline{x}N\overline{\omega}_m + (x_n - \overline{x})\sum_{k=1}^{N} x_k\omega_{km}\right] \tag{A37}$$

or, collecting terms,

$$\overline{\omega}_m + (x_n - \overline{x})\sum_{k=1}^{N}\psi_k\omega_{km} \tag{A38}$$

where ψ_k is the least-squares regression weight:

$$\psi_k = \frac{x_k - \overline{x}}{\sum_{m=1}^{N}(x_m - \overline{x})^2}. \tag{A39}$$

As $\mathbf{\Omega X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ in (A34) is the transpose of $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Omega}$, the typical element of $\mathbf{\Omega X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ is

$$\overline{\omega}_n + (x_m - \overline{x})\sum_{k=1}^{N}\psi_k\omega_{kn}. \tag{A40}$$

What is left is to determine the typical element of $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Omega X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ in (A34). The typical element of $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ is

$$\left(\sum_{k=1}^{N}(x_k - \overline{x})^2\right)^{-1}\left(\frac{1}{N}\sum_{k=1}^{N}x_k^2 - x_n\overline{x} + (x_n - \overline{x})x_m\right) \tag{A41}$$

or

$$\left(\sum_{k=1}^{N}(x_k - \overline{x})^2\right)^{-1}\left(\frac{1}{N}\sum_{k=1}^{N}(x_k - \overline{x})^2 + (x_n - \overline{x})(x_m - \overline{x})\right). \tag{A42}$$

Multiplying $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Omega}$, the typical element of which is given by (A38), with $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, the typical element of which is given by (A42), yields

$$\left(\sum_{p=1}^{N}(x_p - \overline{x})^2\right)^{-1}\left[\sum_{g=1}^{N}\left(\overline{\omega}_g + (x_n - \overline{x})\sum_{k=1}^{N}\psi_k\omega_{kg}\right)\left(\frac{1}{N}\sum_{k=1}^{N}(x_k - \overline{x})^2 + (x_g - \overline{x})(x_m - \overline{x})\right)\right] \tag{A43}$$

as typical element of $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Omega X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. This can be further rewritten as

$$\sum_{g=1}^{N}\left(\overline{\omega}_g + (x_n - \overline{x})\sum_{k=1}^{N}\psi_k\omega_{kg}\right)\left(\frac{1}{N} + \psi_g(x_m - \overline{x})\right) \tag{A44}$$

or as

$$\overline{\omega} + (x_n - \overline{x})\sum_{k=1}^{N}\psi_g\overline{\omega}_g + (x_m - \overline{x})\sum_{k=1}^{N}\psi_g\overline{\omega}_g + (x_n - \overline{x})(x_m - \overline{x})\sum_{k=1}^{N}\sum_{k=1}^{N}\psi_k\psi_g\omega_{kg}. \tag{A45}$$

Collecting terms in (A38), (A40), and (A45) nd using the fact that the typical element of $\mathbf{\Omega}$ in (A34) is ω_{nm} yields that the typical element of the right-hand side of (A34) is

$$\omega_{nm} - \overline{\omega}_n - \overline{\omega}_m + \overline{\omega} - (x_m - \overline{x}) \sum_{k=1}^{N} \psi_k(\omega_{kn} - \overline{\omega}_k) \tag{A46}$$
$$-(x_n - \overline{x}) \sum_{k=1}^{N} \psi_k(\omega_{km} - \overline{\omega}_k) + (x_m - \overline{x})(x_n - \overline{x}) \sum_{k=1}^{N} \sum_{g=1}^{N} \psi_g \psi_k \omega_{kg}.$$

This is identical to (A29). As shown above, rewriting (A29) as (A30) yields the right-hand side of equation (36). Hence, (36) in the main text can be written as $\mathbf{\Pi} = \mathbf{M\Omega M}$.

4 Proof of Proposition 2

To prove the proposition it is useful to define $\phi = \sigma^2/\sigma^2_{US}$. As $0 \leq \sigma^2 < \sigma^2_{US}$, it follows that $\phi \in [0, 1)$. Recall that the two solutions for q in (26) in the main text are β and $\phi(\delta - 1)\beta$, implying $q_1 + q_2 = [1 + \phi(\delta - 1)]\beta$. Hence, the two solutions for q divided by $q_1 + q_2$ are $1/[1 + \phi(\delta - 1)]$ and $\phi(\delta - 1)/[1 + \phi(\delta - 1)]$. This implies that if $\delta \in [0, 2]$, then $\kappa = 1/[1 + \phi(\delta - 1)]$. Hence, using (17) in the main text, $\kappa b = b/[1 + \phi(\delta - 1)] = \beta$.

5 Proof of Proposition 3

For $\delta \in [0, 2]$, see the proof of Proposition 2. To prove it for other values of δ, it is useful to distinguish the cases $\delta > 2$ and $\delta < 0$. We continue to use the definition $\phi = \sigma^2/\sigma^2_{US}$ with $\phi \in [0, 1)$ as $Var(z_i) > 0$ implies that $0 \leq \sigma^2 < \sigma^2_{US}$.

Recall that the two solutions for q in (26) in the main text are β and $\phi(\delta - 1)\beta$, implying $q_1 + q_2 = [1 + \phi(\delta - 1)]\beta$. Hence, the two solutions for q divided by $q_1 + q_2$ are $1/[1 + \phi(\delta - 1)]$ and $\phi(\delta - 1)/[1 + \phi(\delta - 1)]$. Clearly, $1 + \phi(\delta - 1) \geq 0$ for $\delta > 2$. Therefore, the definition of κ in (27) implies

$$\begin{aligned} \kappa &= \tfrac{1}{1+\phi(\delta-1)} \quad \text{if } \phi(\delta - 1) \leq 1 \\ \kappa &= \tfrac{\phi(\delta-1)}{1+\phi(\delta-1)} \quad \text{if } \phi(\delta - 1) > 1. \end{aligned} \tag{A47}$$

Using the notation $\kappa(\phi)$ to capture that κ is a function of ϕ, this can be written as

$$\kappa(\phi) = \begin{cases} \frac{1}{1+\phi(\delta-1)} & \text{if } \phi \in \left[0, \frac{1}{\delta-1}\right] \\ \frac{\phi(\delta-1)}{1+\phi(\delta-1)} & \text{if } \phi \in \left[\frac{1}{\delta-1}, 1\right) \end{cases} \tag{A48}$$

where $0 < 1/(\delta - 1) < 1$. The function $\kappa(\phi)$ is illustrated in Figure A1. $\kappa(\phi)$ is strictly decreasing in ϕ up to the point where $\phi = 1/(\delta - 1) < 1$, and is strictly increasing in ϕ from

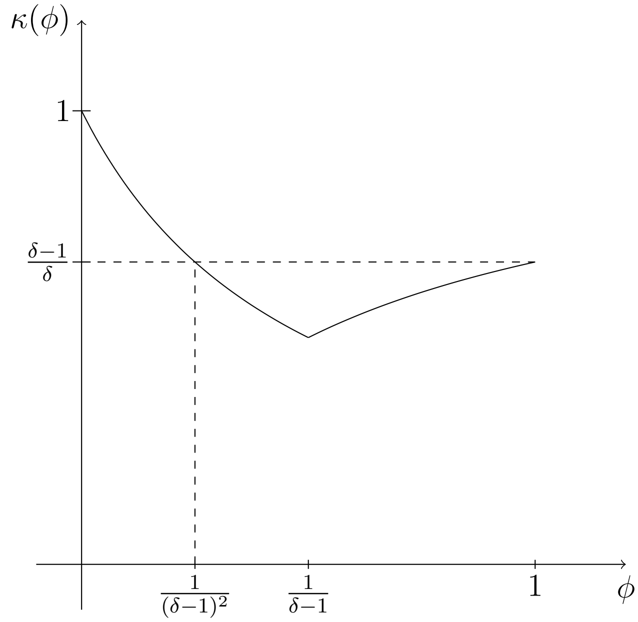

Figure A1: The shape of $\kappa(\phi)$ for $\delta > 2$.

that point on. Moreover, $\kappa(1) = (\delta - 1)/\delta$. As $\kappa(\phi)$ is strictly increasing for $\phi > 1/(\delta - 1)$, we get that $\kappa(\phi) < (\delta - 1)/\delta$ for all $\phi \in [1/(\delta - 1), 1)$.

For $\delta > 2$, the relevant version of condition (28) in Proposition 3 is

$$\kappa \geq \frac{\delta - 1}{\delta}. \tag{A49}$$

It can therefore never be satisfied for $\phi \in (1/(\delta-1), 1)$. Put differently, the relevant condition in the proposition can be satisfied only if $\phi \in [0, 1/(\delta-1)]$. For ϕ in this range, (A48) implies $\kappa(\phi) = 1/[1+\phi(\delta-1)]$ and the condition in (A49) is satisfied if $\phi \leq 1/(\delta-1)^2$. Summarizing, when $\delta > 2$, the relevant condition in Proposition 3 is satisfied if and only if ϕ satisfies

$$\phi(\delta - 1)^2 \leq 1. \tag{A50}$$

As $\kappa = 1/[1 + \phi(\delta - 1)]$ for ϕ in this range, the claim $\beta = \kappa b$ in Proposition 3 follows from rewriting (17) in the main text as $b = [1 + \phi(\delta - 1)]\beta$.

When $\delta < 0$, the two solutions for q divided by $q_1 + q_2$, $1/[1+\phi(\delta-1)]$ and $\phi(\delta-1)/[1+\phi(\delta-1)]$, imply that κ in Proposition 3 is

$$\begin{array}{ll} \kappa = \frac{1}{1+\phi(\delta-1)} & \text{if } \phi(\delta-1) \geq -1 \\ \kappa = \frac{\phi(\delta-1)}{1+\phi(\delta-1)} & \text{if } \phi(\delta-1) < -1 \end{array}. \tag{A51}$$

Or, using the notation $\kappa(\phi)$ to capture that κ is a function of ϕ:

$$\kappa(\phi) = \begin{cases} \frac{1}{1+\phi(\delta-1)} & \text{if } \phi \in \left[0, -\frac{1}{\delta-1}\right] \\ \frac{\phi(\delta-1)}{1+\phi(\delta-1)} & \text{if } \phi \in \left[-\frac{1}{\delta-1}, 1\right) \end{cases} \tag{A52}$$

where $0 < -1/(\delta-1) < 1$. The function $\kappa(\phi)$ is illustrated in figure A2. For $\phi < -\frac{1}{\delta-1}$,

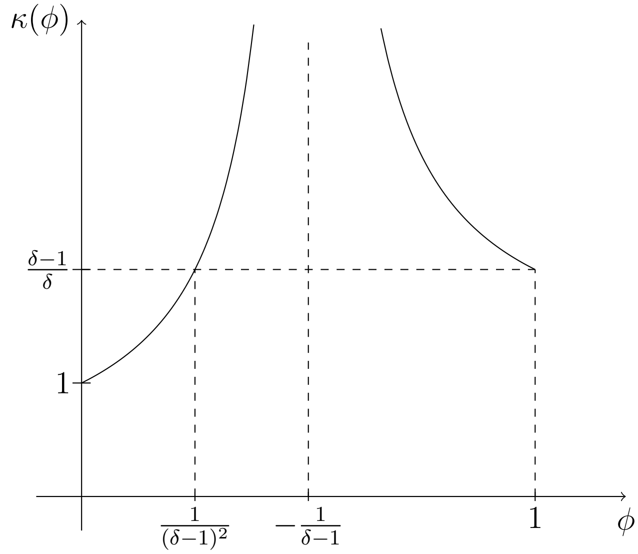

Figure A2: The shape of $\kappa(\phi)$ for $\delta < 0$.

κ is strictly increasing in ϕ. For values of ϕ larger than $\phi = -1/(\delta-1)$, $\kappa(\phi)$ is strictly decreasing. Furthermore, $\kappa(1) = (\delta-1)/\delta$. As a result, we get that $\kappa(\phi) > (\delta-1)/\delta$ for $\phi \in (-1/(\delta-1), 1)$. For $\delta < 0$, the relevant version of condition (28) is

$$\kappa \leq \frac{\delta-1}{\delta}. \tag{A53}$$

For $\phi \in (-1/(\delta-1), 1)$, it can never be satisfied. Put differently, the condition in (A52) can be satisfied only if $\phi \in [0, -1/(\delta-1)]$. For ϕ in this range, (A51) implies $\kappa = 1/[1+\phi(\delta-1)]$ and hence that (A52) is satisfied if $\phi(\delta-1)^2 \leq 1$. Summarizing, when $\delta < 0$, the condition in Proposition 3 is satisfied if and only ϕ satisfies

$$\phi(\delta-1)^2 \leq 1. \tag{A54}$$

As we have $\kappa = 1/[1+\phi(\delta-1)]$ for ϕ in this range, the claim $\beta = \kappa b$ in Proposition 3 follows from rewriting (17) in the main text as $b = [1+\phi(\delta-1)]\beta$ once again.

It remains to be shown that if the condition in Proposition 3 is not satisfied, then the parameters b, η, and δ do not allow us to determine which of the two solutions for q in (26) in the main text identifies β. Consider first the case $\delta > 2$. In this case, κ as defined in (27) is given by (A48). To capture that κ in (A48) is a function of ϕ, we use the notation $\kappa(\phi)$. It is straightforward to establish that if $\delta > 2$ and the value of κ' implied by the two solutions for q in (26) satisfies $\kappa' < (\delta - 1)/\delta$ – that is, the condition relevant for the case $\delta > 2$ in Proposition 3 (also stated in (A49)) is not satisfied – then the equation $\kappa(\phi) = \kappa'$ has two solutions for ϕ that satisfy $\phi \in [0, 1)$. Moreover, one of the two solutions for ϕ is smaller than $1/(\delta - 1)$ and the other solution for ϕ is larger than $1/(\delta - 1)$. As a result, $\beta = \kappa b$ for one of the solutions (the solution for ϕ smaller $1/(\delta - 1)$) and $\beta = (1 - \kappa)b$ for the other solution. As both solutions for q in (26) are consistent with the parameters b, η, and δ, and both solutions yield that the implied ϕ satisfies $\phi \in [0, 1)$, it is impossible to know which of the two solutions for q in (26) identifies β. The proof for the case $\delta < 0$ is analogous.

6 Proof of Proposition 4

In proving Proposition 3 we have shown that the condition in (28) holds if and only if $(\delta - 1)^2\sigma^2/\sigma_{US}^2 \leq 1$.

7 Proof of Proposition 5

From Proposition 4, we know that the condition in (28) is not satisfied if and only if $\phi(\delta - 1)^2 > 1$. In these circumstances we only know that β is one of the two solutions for q in (26), that is $\beta \in \{q_1, q_2\}$. As $q_1 + q_2 = b$, this implies that $\beta/b \in \{q_1/(q_1 + q_2), q_2/(q_1 + q_2)\}$. Or, making use of the definition for κ in (27) in the main text, $\beta/b \in \{\kappa, 1 - \kappa\}$.

When $\delta > 2$, it follows from (A47) that for $\phi(\delta - 1)^2 > 1$ or, equivalently, for $\phi \in (1/(\delta - 1)^2, 1)$: $\kappa < (\delta - 1)/\delta$. This in turn implies that $1 - \kappa > 1/\delta$. As $(\delta - 1)/\delta > 1/\delta$ when $\delta > 2$, it follows that $\beta/b \in \{\kappa, 1 - \kappa\}$ implies $\beta/b \in (1/\delta, (\delta - 1)/\delta)$. This establishes the part of the proposition that applies to $\delta > 2$.

When $\delta < 0$, it follows from (A51) that for $\phi(\delta - 1)^2 > 1$ or, equivalently, for $\phi \in (1/(\delta - 1)^2, 1)$: $\kappa > (\delta - 1)/\delta$. This in turn implies that $1 - \kappa < 1/\delta$. As $(\delta - 1)/\delta > 1/\delta$ when $\delta < 0$, it follows that $\beta/b \in \{\kappa, 1 - \kappa\}$ implies $\beta/b \notin [1/\delta, (\delta - 1)/\delta]$. This establishes the part of the proposition that applies to $\delta < 0$.